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The stability of equilibrium in systems with friction☆

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Abstract

Mechanical systems with non-ideal geometrical constraints are considered. The possible lack of uniqueness of the solution of the problem of determining the generalized accelerations and reactions with respect to specified coordinates and velocities is taken into account in solving the problem of the stability of an equilibrium state. A number of necessary and sufficient conditions of stability are obtained. It is shown that the results are also applicable in the case of unilateral constraints subject to the condition that a specific hypothesis concerning the character of the impacts on the constraints is adopted. A problem on the stability of a rigid body on a rough plane in the two-dimensional case is solved as an example.

This problem has been investigated previously under the assumption that the normal stresses at points of frictional contact are determined.[1–3] Such a formulation is not exhaustive since, in systems with friction, cases of non-regularity are possible when it is impossible to determine the motion uniquely.[4] In particular, besides equilibrium, the onset of motion is possible.[5] The sufficient conditions for regularity have been obtained.[6]

1. Formulation of the problem

Using the principle of release from constraints, we write the equations of motion in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q} + \mathbf{R}, \quad \mathbf{q}, \mathbf{Q}, \mathbf{R} \in R^n \tag{1.1}$$

where T is the kinetic energy and $\mathbf{Q}$ are generalized forces. Without loss of generality, we shall assume that T is a quadratic form in the generalized velocities $\dot{\mathbf{q}}$: the existence of terms of the first and zero powers can be taken into account when determining the generalized forces.

In accordance with the adopted hypothesis concerning the friction law, the reactions of the constraints R satisfy the system of equations

$$f(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{R}) = 0, \quad \mathbf{f} \in R^m \tag{1.2}$$

The existence in formulae (1.2) of a dependence on the generalized accelerations is explained by the nature of dry friction: if the generalized velocities are equal to zero (that is, the motion of the system begins at a given instant of time), the direction of the friction forces is, in fact, determined by the accelerations.

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System (1.1), (1.2) can be incompatible or indeterminate. Examples of such situations were discovered earlier in the case of Coulomb friction.[4] We shall say that this system is regular at a given point of the phase space $(\mathbf{q}, \dot{\mathbf{q}})$, if the variables $\mathbf{R}$, $\ddot{\mathbf{q}}$ are uniquely defined. In the general case, on solving the system for the generalized accelerations, we arrive at an inclusion of the form

$$\ddot{\mathbf{q}} \in \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{F} \in R^n \tag{1.3}$$

where the set of values of the vector function $\mathbf{F}$ can be empty, finite or infinite.

At a point of regularity, the function $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}})$ is uniquely defined. Note that this does not guarantee only continuity but even the uniqueness of this function in the neighbourhood of this point. In particular, the use of Coulomb's law leads to velocity discontinuities on transition from rest to sliding. In the case of unilateral constraints, the reactions can be of an impulsive (impact) nature.

The property of regularity is a key property when investigating the stability of equilibrium since the existence of other motions for given initial conditions is directly indicative of the instability of the equilibrium. If, however, this property is satisfied, it is possible to use the known results regarding the stability of the solutions of differential inclusions.[7]

Usually in problems of mechanics concerned with finding equilibrium positions, it is checked that the compensating external forces satisfy the law (1.2) (see Ref. 8). In order to draw a conclusion concerning regularity, it is additionally necessary to establish the impossibility of other motions. To do this, it is necessary to take an arbitrary vector of the virtual accelerations $\delta\dot{\mathbf{q}}$ and to consider the system of equations (1.1), (1.2). Its incompatibility for all $\delta\ddot{\mathbf{q}} \neq 0$ implies the regularity of the inclusion (1.3) at the point $(\mathbf{q}^*, 0)$. An analogous approach has been used earlier in a problem concerning the equilibrium of a rigid body on a rough plane.[9]

Example. We will consider the plane problem of the equilibrium of a body with one of its points C touching a rough plane. We shall assume that the external forces have a resultant passing through the point C, since equilibrium is otherwise impossible. We direct the abscissa along the support and take $\mathbf{q} = (x_C, y_C, \varphi)$, where x_C, y_C are the coordinates of the contact point and φ is the angle between the radius vector $GC = (a, -h)$ (G is the centre of mass of the body) and the OX axis. The main theorems of dynamics, when account is taken of the initial condition $\dot{\mathbf{q}} = 0$, are expressed by the equations (we assume the mass to be a unit mass)

$$\ddot{x}_C = X + R_x + h\ddot{\varphi}, \quad \ddot{y}_C = Y + R_y + a\ddot{\varphi}, \quad k^2\ddot{\varphi} = a(Y + R_y) + h(X + R_x) \tag{1.4}$$

where X, Y and R_x, R_y are the projections of the external force and the reaction of the plane onto the coordinate axes and k is the radius of gyration. The quantity h is always positive and, without loss of generality, we shall also assume that $a > 0$.

It follows from formulae (1.4) that

$$\begin{gathered} \ddot{x}_C = a_{11}(X + R_x) + a_{12}(Y + R_y), \quad \ddot{y}_C = a_{12}(X + R_x) + a_{22}(Y + R_y) \\ a_{11} = 1 + h^2/k^2, \quad a_{12} = ah/k^2, \quad a_{22} = 1 + a^2/k^2 \end{gathered} \tag{1.5}$$

The two Eq. (1.5) contain four unknown quantities (accelerations and reactions) which are related by the inequality $\ddot{y}_C \geq 0$ and Coulomb's law

$$\begin{gathered} \ddot{y}_C > 0 \Rightarrow R_x = R_y = 0 \\ \ddot{x}_C = 0 \Rightarrow |R_x| \leq \mu R_y, \quad \ddot{x}_C \neq 0 \Rightarrow R_x = -s\mu R_y, \quad s = \operatorname{sign}\ddot{x}_C \end{gathered} \tag{1.6}$$

where μ is the coefficient of friction. In order to solve system (1.5), (1.6), it is necessary to consider all possible forms of motion: detachment of the body from the support and the onset of sliding to the left or to the right.

1°. When there is separation $\ddot{y}_C > 0$, $R_x = R_y = 0$. This type of motion is possible if

$$a_{12}X + a_{22}Y > 0 \tag{1.7}$$

2°. At equilibrium $R_x = -X$, $R_y = -y$ and, in accordance with relations (1.6),

$$Y \le 0, \quad |X| \le \mu|Y| \tag{1.8}$$

3°. In the case of sliding to the left $\ddot{y}_C = 0, \ddot{x}_C < 0, s = -1$. Expressing R_y from the second equation of (1.5) and substituting the value found into the first equation, we obtain $\ddot{x}_C > 0$. The contradiction confirms the impossibility of this type of motion.

4°. In the case of sliding to the right, $\ddot{y}_C = 0, \ddot{x}_C > 0, s = 1$. When account is taken of relations (1.6), from the second equation of (1.5) we obtain

$$R_y = -\frac{a_{12}X + a_{22}Y}{a_{22} - \mu a_{12}}, \quad \ddot{x}_C = \Delta\frac{X + \mu Y}{a_{22} - \mu a_{12}}; \quad \Delta = a_{11}a_{22} - a_{12}^2 > 0 \tag{1.9}$$

When conditions (1.8) are satisfied, the numerator in the expression for $\ddot{x}_C$ is negative and the type of motion being considered is therefore possible if $R_y > 0$, $a_{22} < \mu a_{12}$, that is, if the inequality (1.7) and, also,

$$a^2 + k^2 < \mu a h \tag{1.10}$$

are satisfied.

Moreover, if $a_{22} = \mu a_{12}$ and $X = -\mu Y$, then the second equation of (1.5) becomes an identity and, at the same time, $\ddot{x}_C = -\mathbf{\Delta}(Y + R_y)/a_{12}$. This means that system (1.5), (1.6) has an infinite number of solutions for which $R_y < -Y$.

We will now sum up. Inequalities (1.8) express the conditions of equilibrium. If inequality (1.7) is satisfied, then, when (1.8) is taken into account, condition (1.10) also holds. Here, in the system, together with equilibrium, separation from the support is also possible as well as sliding to the right. If the relations obtained from (1.7) and (1.10) are satisfied by replacing the inequality signs by an equality, then, together with equilibrium, there is a continuum of solutions, for which the body slides to the right, which differ in the magnitude of the reaction and acceleration.

The conditions for the regularity of the equilibrium consist of relations (1.8) and the inequality

$$ahX + (a^2 + k^2)Y < 0 \tag{1.11}$$

It can be shown that these conditions guarantee regularity in the neighbourhood of equilibrium. To do this, it is necessary to take account of the fact that, as the coordinates change, the parameters a and h vary continuously, and, when there is sliding which arises for non-zero generalized velocities, the paradoxical situations of non-existence or non-uniqueness, known as the Painlevé paradoxes, do not arise.

We will illustrate the results in the plane of the parameters $\xi = X/Y$ and μ (Fig. 1). The equilibrium domain is a square bounded by the bisectors of the first and second coordinate angles. The part of this domain lying to the right of the line $\xi = \xi_1$, $\xi_1 = -(k^2 + a^2)/(ah)$ (the cross-hatched area) corresponds to regular equilibrium (the two domains are identical in the case when $a = 0$). For values of the parameters in the sector between this line and the bisector of the second coordinate angle (the singly hatched area) the equations of motion have three solutions, and a continuum of solutions corresponds to the vertex of the sector S.

The case when the constraint between the body and the support is bilateral and separation is impossible is treated in a similar manner. In this case, $\ddot{y}_C \equiv 0$ and the normal component of the reaction R_y can have any sign. The equilibrium condition reduces to the second inequality of (1.8) and the condition of regularity reduces to an inequality which is

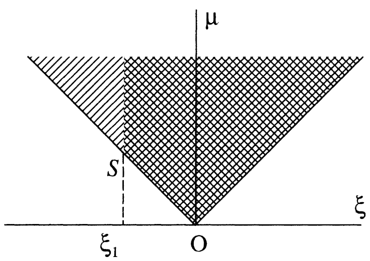

Fig. 1.

the opposite of (1.7) (if $Y<0$) or to inequality (1.7) (if $Y>0$). Outside the domain of regularity the system also has two solutions in addition to the equilibrium solution and both correspond to sliding to the right (if $Y<0$) or to the left (if $Y>0$).

Remark 1. A comprehensive analysis of the equilibrium conditions has been carried out in the example considered above. In more complex problems, such an analysis is made more difficult in view of the abundance of possible forms of motion. In this case, some of the possible forms can be discarded by making use of simple energy considerations. Since the kinetic energy is zero for the given initial conditions, motions for which the sum of the work of the external forces and friction is negative are impossible and, therefore, instead of the inconsistency of system (1.1), (1.2), it is sufficient to prove the inequality

$$(R+F)\delta\ddot{q} \le 0 \tag{1.12}$$

This approach was used in Ref. 10 to prove of Hill stability. It should be noted that requirement that inequality (1.12) is satisfied for all virtual accelerations is too restrictive in systems with unilateral constraints. In the example considered above in the case when $a=0$, the regularity conditions are satisfied. However, inequality (1.12) is violated for certain releasing displacements (if $X \neq 0$).

2. Stability conditions in the case of bilateral constraints

It has been mentioned that, generally speaking, it is impossible to describe a perturbed motion using ordinary differential equations with a smooth right-hand side in a system with friction, and we shall therefore apply the concept of stability directly to the equilibrium of the mechanical system. Note that, in systems with friction, the equilibrium positions are not isolated as a rule and, therefore, the formulations of a problem regarding the stability of a given equilibrium position or concerning the asymptotic stability of a family of equilibria are meaningful. In this section, system (1.1) with bilateral constraints is considered.

Proposition 1. *Regularity of the system at the point* $(\mathbf{q}^*, 0)$ *is necessary for the equilibrium* $\mathbf{q}^*$ *to be stable.*

Actually, the existence of motions which are different from the equilibrium in the case of initial zero perturbations is a direct indication of instability.

Proposition 2. *Suppose system (1.1) is regular in a certain neighbourhood of the equilibrium position* $(\mathbf{q}^*, 0)$ *and, moreover, the generalized forces are continuous with respect to* $(\mathbf{q}, \dot{\mathbf{q}})$ *and the reactions of the constraints are continuous with respect to* $\mathbf{q}$. *If the conditions*

$$(Q_i(\mathbf{q}^*, 0) + R_i(\mathbf{q}^*, \dot{\mathbf{q}}))\dot{\mathbf{q}}_i \le -\delta|\dot{\mathbf{q}}_i|, \quad i = 1, \dots, n \tag{2.1}$$

are satisfied for a certain number $\delta > 0$ *and sufficiently small values of* $||\dot{\mathbf{q}}||$, *then the equilibrium is stable and, what is more, a perturbed motion ceases over a finite time.*

Proof. According to the theorem on a change in the kinetic energy

$$\frac{dT}{dt} = \sum_{i=1}^{n} (Q_i(\mathbf{q}, \dot{\mathbf{q}}) + R_i(\mathbf{q}, \dot{\mathbf{q}}))\dot{q}_i$$

whence, for sufficiently small values of $||\mathbf{q} - \mathbf{q}^*||$ and $||\dot{\mathbf{q}}||$, we have

$$\frac{dT}{dt} = \sum_{i=1}^{n} (Q_i(\mathbf{q}^*, \mathbf{0}) + R_i(\mathbf{q}^*, \dot{\mathbf{q}}) + \alpha(\mathbf{q} - \mathbf{q}^*, \dot{\mathbf{q}}))\dot{q}_i \tag{2.2}$$

where α is an infinitesimal. Taking account of conditions (2.1), we obtain

$$\frac{dT}{dt} \le -\delta_1 \sum_{i=1}^{n} |\dot{q}_i|, \quad 0 < \delta_1 \le \delta \tag{2.3}$$

Since T is a quadratic form in $\dot{q}$ and, in view of the equivalence of norms in R^n, the inequality

$$\frac{dT}{dt} \le -c\sqrt{T}, \quad c > 0 \tag{2.4}$$

follows from relations (2.3).

We shall assume that $T > 0$ when $t \in [t_0, t_1]$. Integrating inequality (2.4), we obtain

$$\sqrt{T_1} \le \sqrt{T_0} - c(t_1 - t_0)/2 \tag{2.5}$$

When $t_1 > t_0 + 2\sqrt{T_0}/c$, the right-hand side of inequality (2.5) is negative, which indicates stopping of motion. Hence, perturbed motion occurs with a velocity of the order of $\dot{q}_0$ and ceases after a time of the same order. The overall change in the coordinates tends to zero together with the perturbations, which also proves the stability. □

Remark 2. This assertion has been proved earlier in the case of Coulomb friction.[1] Here, inequalities (2.1) mean that the system is completely dissipative and, moreover, at each of the points of frictional contact, the resultant force lies in the "stagnation" zone, that is, within the friction cone. It is important to take account of the discontinuous form of relation (1.3) in the neighbourhood of an equilibrium. In view of this, the generalized accelerations do not tend to zero when $\mathbf{q} \to \mathbf{q}^*$, $\dot{\mathbf{q}} \to 0$, which confirms the existence of considerable inertial forces leading to a redistribution of the normal load. This fact had earlier remained unnoticed (see Refs. 1,9,11), that is, the normal reactions were assumed to be unchanged.

Proposition 3. *We shall assume that system (1.1), (1.2) is regular at the point* $(\mathbf{q}^*, 0)$ *but that the reactions of the constraints are non-uniquely defined in the neighbourhood of this point: there is a family* $\mathbf{R}^\lambda(\mathbf{q}, \dot{\mathbf{q}})$, $\lambda \in \Lambda$ *which satisfies the system. Suppose the generalized forces are continuous in* $\mathbf{q}$, $\dot{\mathbf{q}}$ *and the reactions are equicontinuous in* $\mathbf{q}$, *that is,*

$$\left\|\mathbf{R}^\lambda(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{R}^\lambda(\mathbf{q}^*, \dot{\mathbf{q}})\right\| \le \beta(\mathbf{q} - \mathbf{q}^*)$$

where $\beta \to 0$ *when* $\mathbf{q} \to \mathbf{q}^*$. *If conditions (2.1) are satisfied for a certain number* $\delta > 0$, *fairly small values of* $\dot{\mathbf{q}}$ *and all possible values of the reactions, then the assertions of* Proposition 2 *hold.*

Proof. According to the condition, the relation

$$\frac{dT}{dt} = \sum_{i=1}^{n} (Q_i(\mathbf{q}^*, \mathbf{0}) + R_i^\lambda(\mathbf{q}^*, \dot{\mathbf{q}}) + \alpha(\mathbf{q} - \mathbf{q}^*, \dot{\mathbf{q}}))\dot{q}_i$$

is satisfied for every $\lambda \in \Lambda$. The subsequent reasoning is similar to the proof of Proposition 2.

This assertion can be used to investigate systems with a continual domain of contact in which the unique determination of the distribution of the normal load is impossible without invoking additional hypotheses.

We will now consider systems with incomplete dissipation. In this case, in system (1.1) some of the reactions are identically equal to zero:

$$R_1 = R_2 = \ldots = R_k \equiv 0, \quad 0 < k < n \tag{2.6}$$

It is necessary that the corresponding generalized forces also vanish at the equilibrium position, that is,

$$Q_1(\mathbf{q}^*, \mathbf{0}) = \ldots = Q_k(\mathbf{q}^*, \mathbf{0}) = 0$$

We will now define an auxiliary system with k degrees of freedom, fixing the values of the coordinates in system (1.1) for which any change is accompanied by friction:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \quad q_s \equiv \bar{q}_s; \quad j = 1, \ldots, k; \quad s = k+1, \ldots, n \tag{2.7}$$

Proposition 4. *Suppose system (2.7) has an equilibrium position* $q_j = q_j^*(\bar{\mathbf{q}})(j = 1, \ldots, k)$, *which is uniformly stable with respect to* $\bar{\mathbf{q}}$, *for any values* $\bar{q}_s$ *from a certain neighbourhood of the point* $q_s^*(x = k+1, \ldots, n)$, (*that is, in the definition of stability the relation* $\delta(\varepsilon)$ *is the same for all* $\bar{\mathbf{q}}$). *If the conditions of* Proposition 2 or Proposition 3 *are satisfied for the variables with the subscripts* $i = k+1, \ldots, n$, *then the equilibrium* $\mathbf{q}^*$ *of system (1.1) is also stable.*

Proof. We will show that, after a finite time, which vanish together with the initial perturbations, the values of the coordinates q_s are fixed. The subsequent motion will be described by system (2.7) which, according to the condition, is stable. We now define generalized impulses from the formulae $p_j = \partial T/\partial \dot{q}_j$ $(j = 1, \ldots, k)$ and transform system (1.1) to the Routh form

$$\frac{dq_j}{dt} = -\frac{\partial\Phi}{\partial p_j}, \quad \frac{dp_j}{dt} = \frac{\partial\Phi}{\partial q_j} + Q_j, \quad \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{q}_s} - \frac{\partial\Phi}{\partial q_s} = Q_s + R_s$$
$$j = 1, \ldots, k; \quad s = k+1, \ldots, n \tag{2.8}$$
$$\Phi(\mathbf{q}, \dot{q}_s, p_j) = T - \sum_{j=1}^{k} p_j \dot{q}_j$$

It can be shown that the function Φ decomposes into the sum of two quadratic forms which depend on p_j and $\dot{q}_s$. In view of this, the second group of equations of (2.8) does not contain derivatives of the generalized impulses. On applying the methods used to prove Propositions 2 and 3 to this subsystem, we arrive at the above-mentioned conclusion concerning the cessation of motion. The subsequent motion is described by the first (Hamiltonian) group of Eq. (2.8) for fixed $\mathbf{q}_s = \bar{\mathbf{q}}_s$, that is, it has a stable form. □

Remark 3. A similar assertion has been proved earlier in the special case when the generalized forces Q_j admit of a force function.[1]

In order to illustrate the results obtained above, we will consider a number of examples of the investigation of the stability of a rigid body which is in contact with a rough plane (a bilateral constraint). We shall assume that the motion occurs in a fixed plane, perpendicular to the support, and we will denote the coefficient of friction by μ. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ form a basis in this plane, where $\mathbf{j}$ is normal to the support. The centre of mass of the body is located at the point G, which is a distance h from the support in the direction of the unit vector $\mathbf{j}$. The contact point C_k is characterized by the projection of a_k of the vector $\mathbf{GC}_k$ onto the direction of the unit vector $\mathbf{i}$.

Examples. 1°. The body touches the plane at two points: C_1 and C_2. The external forces reduce to the vectors $Y_1\mathbf{j}$ and $Y_2\mathbf{j}$, applied at these points, and the tangential component $X\mathbf{i}$ is directed along the straight line C_1C_2. In accordance with Coulomb's law, the body can be in equilibrium if the following inequality is satisfied

$$|X| \le \mu(|Y_1| + |Y_2|) \tag{2.9}$$

In order to verify the necessary condition of stability (Proposition 1), it has to be convincingly demonstrated that it is impossible for sliding to start from a state of rest with an acceleration $\ddot{x}$. If it does take place, then an inertial force $I = -m\ddot{x}$, applied to the centre of mass, is added to the system. This force decomposes into the sum of a vector I acting along C_1C_2 and a couple with a moment $M = -hI$. We bring this couple to the points C_1 and C_2 by imposing normal forces $\mp M/a$, $a = ||C_1C_2|$ on it. D'Alembert's principle is expressed by the equality

$$z = X - \bar{\mu}(|z - \bar{Y}_1| + |z + \bar{Y}_2|)\operatorname{sign} z; \quad z = m\ddot{x}, \quad \bar{Y} = \frac{a}{h}Y, \quad \bar{\mu} = \frac{h}{a}\mu \tag{2.10}$$

which is an algebraic equation in z (the second term on the right-hand side corresponds to the friction forces). Without loss of generality, we will assume that $X > 0$ and Eq. (2.10) can then only have positive roots.

The right-hand side of Eq. (2.10) is a trilinear function which is negative when $z = 0$ (as a consequence of condition (2.9)) and is only an increasing function in the interval $0 \le z \le \min\{\bar{Y}_1, -\bar{Y}_2\}$ Consequently, if Eq. (2.10) has positive roots, then $Y_1 > 0$, $Y_2 < 0$ and one of the roots lies in this interval and satisfies the equation

$$(1 - 2\bar{\mu})z = X - \mu(|Y_1| + |Y_2|) \tag{2.11}$$

the right-hand side of which is positive. If condition (2.9) is satisfied as an equality, then, when $\bar{\mu} = 1/2$, Eq. (2.11) becomes an identity. Otherwise, for the existence of a positive root, it is necessary that $\bar{\mu} = 1/2$. In addition,

$$X - \bar{\mu}(|\bar{Y}_1| + |\bar{Y}_2|) \ge (1 - 2\bar{\mu})\min\{|\bar{Y}_1|, |\bar{Y}_2|\} \tag{2.12}$$

Elementary analysis shows that this requirement is equivalent (when account is taken of condition (2.9)) to the following system of inequalities

$$Y_1 > 0, \quad Y_2 < 0, \quad X \geq (\bar{Y}_1 - \bar{Y}_2)/2 + (\bar{\mu} - 1/2)\left|\bar{Y}_1 + \bar{Y}_2\right|$$

The case when $X < 0$ is considered in a similar manner. As a result, we obtain the condition for the regularity of the equilibrium: either the numbers Y_1 and Y_2 have the same sign or $\bar{\mu} \leq 1/2$, or the following inequality is satisfied

$$|X| < \left|\bar{Y}_1 - \bar{Y}_2\right|/2 + (\bar{\mu} - 1/2)\left|\bar{Y}_1 + \bar{Y}_2\right| \tag{2.13}$$

Note that condition (2.13) includes not only the dynamic but, also, the geometrical parameters of the system and it therefore does not follow from the condition for the possibility of equilibrium (2.9). In particular, if $Y_1 = -Y_2$, then the domain (2.13) consists of half of the domain (2.9).

In order to explain the meaning of inequality (2.1), it is necessary to take account of Remark 2: sliding of a body leads to a redistribution of the normal load. Suppose N_1 and N_2 are the normal reactions at the contact points. Then, the total friction force is equal to

$$R_x = -\mu(|N_1| + |N_2|)\operatorname{sign}\dot{x}$$

In order to determine N_1 and N_2, we set up a system of equations which expresses the fact that there is no motion in the normal direction:

$$\sum_{k=1}^{2}(Y_k + N_k) = 0, \quad \sum_{k=1}^{2}a_k(Y_k + N_k) + h(X + R_x) = 0 \tag{2.14}$$

We put $z = a(Y_1 + N_1)/h$. Then, it follows from the first equation that $a(Y_2 + N_2) = -hz$, and the second equation (in the case when $\dot{q} > 0$) takes the form of (2.10), When the regularity conditions are satisfied, this equation can only have a negative root: $z_+ < 0$. The analogous equation in the case when $\dot{q} < 0$ has a positive root: $z_- > 0$. On putting $\delta = \min\{|z_+|, |z_-|\}$ in condition (2.1), we obtain that the regularity condition (2.13) is not only necessary but also sufficient for stability.

2°. We shall now assume that the contact of the body with the support occurs at all points of the interval $[C_1, C_2]$ and that the external forces are the same as in the preceding example. In order to determine the possibility of an equilibrium, we will make use of the concept of a guaranteed equilibrium:[12] for any distribution of the normal load in the contact area, the friction forces are sufficient for the system to be at rest. At equilibrium the relations

$$Y_1 + Y_2 + \sum N_i = 0, \quad a_1Y_1 + a_2Y_2 + \sum x_iN_i = 0, \quad \mu\sum|N_i| \geq |X| \tag{2.15}$$

must be satisfied, where $x_i \in [a_1, a_2]$ is the point of application of the normal load N_i, and the total number of such points can be finite or infinite (in the case of a continuous distribution, the sums are replaced by integrals). In the case of a guaranteed equilibrium, inequality (2.5) holds for any $\{N_i\}$ which satisfy equalities (2.15).

In order to check this requirement, as well as the stability conditions (Proposition 3), we will first prove the following assertion: if the numbers N_i satisfy the two Eq. (2.15), then

$$\min\sum|N_i| = |Y_1| + |Y_2|$$

Actually, this equality is attained when $N_1 = -Y_1$, $N_2 = -Y_2$. If the numbers Y_1 and Y_2 have the same sign, the assertion immediately follows for the first equation of (2.15). Next, suppose $Y_1\, Y_2 < 0$. We multiply the first equation of (2.15) by $a_1 + a_2$ and subtract the second equation twice from the result. On then dividing the resulting equality by a, we have

$$\sum\xi_iN_i = Y_2 - Y_1, \quad \xi_i = (a_1 + a_2 - 2x_i)/a \in [-1, 1]$$

Consequently,

$$\sum|N_i| \geq \sum|\xi_i||N_i| \geq \left|\sum\xi_i N_i\right| = |Y_2 - Y_1| = |Y_1| + |Y_2|$$

as was stated.

It follows from this that the conditions of guaranteed equilibrium have the form (2.9). What is more, when investigating the stability conditions (Propositions 1 and 3), we also arrive at a system of equations of the form of (2.15) but with the addition of inertial forces (see (2.10) and (2.14)). The conclusion is: the stability conditions in this example are identical to the analogous conditions (2.13) for the first example.

3°. A body which comes into contact with a support at a unique point possesses two degrees of freedom: it can slide and/or roll (we neglect rolling friction). As the generalized coordinates we can take x_C and φ (see the example in Section 1) since $y_C \equiv 0$. The external forces must be specified in the neighbourhood of the equilibrium in such a way that the equilibrium is possible for all x_C from a certain interval, that is, for a certain value of $\varphi(x_C)$, they reduce to a resultant, passing through the point of contact. It is obvious that the geometrical parameters a and h solely depend on φ and, moreover, in a continuous manner. The conditions for the regularity of the family of equilibria are therefore expressed by inequalities (1.8) and (1.11), where X and Y are calculated when $\varphi = \varphi(x_C)$, $\dot{x}_C = 0$, $\dot{\varphi} = 0$. The conditions of Proposition 4 imply that inequalities (2.1) are satisfied for small $\dot{x}_C$ and fixed φ and there is also uniform sliding when there is no slipping. The first of these conditions is satisfied if inequality (1.8) is strict, and verification of the second condition for the specified forces is feasible using standard methods. In particular, if a heavy body is located on an inclined plane, then, at equilibrium, the of mass and the contact point are located on a single vertical. The stability conditions are satisfied if the tangent of the angle of inclination of the plane is less than the coefficient of friction, and the centre of mass is raised from the equilibrium value in the case of small moves in the position. The "van'ka-vstan'ka" toy (a doll with a weight attached to the base) on an inclined plane is a physical example of a stable equilibrium.

3. The case of unilateral constraints

In the treatment of systems with unilateral constraints, it is necessary, in order to describe the motion, to add certain rules for calculating the impulses to the usual equations of mechanics which is associated with the assumption of additional hypotheses. The set of such hypotheses depends on the physical properties of the system being investigated and can affect the conclusion concerning stability.

Example. Suppose the system

$$\ddot{x} = X - \mu N \operatorname{sign}\dot{x}, \quad \ddot{y} = -Y + N; \quad y \geq 0, \quad 0 \leq X < \mu Y \tag{3.1}$$

is given, where $N \geq 0$ is the normal reaction of the unilateral constraint (similar equations describe the dynamics of a heavy particle on an inclined plane). System (3.1) possesses a family of equilibrium positions $x = x^* \in R$, $y = 0$. By virtue of Proposition 2, if $y \equiv 0$, then each of these equilibria is stable. We shall discuss the stability taking account of the possibility of the weakening of a constraint for different impact models.

The classical theory of impact is based on two hypotheses. The normal components of the relative velocity of the colliding bodies before and after the impact are connected by the relation

$$\dot{y}^+ = -e\dot{y}^-, \quad e \in [0, 1] \tag{3.2}$$

and the components of the impulse satisfy the inequality

$$|I_x| \leq \mu I_y \tag{3.3}$$

Here, the equality sign applies in the case when the tangential component of the velocity after impact preserves its sign and the inequality sign corresponds to the stopping of sliding.

If the Newtonian coefficient of restitution e in formula (3.2) is assumed to be constant and less than unity, then the change in the y coordinate is described by a sequence of flights and impacts and, also, the interval between impacts and the impulses form a progression with a ratio e. Consequently, after a time which tends to zero together with the initial perturbations, the identity $y \equiv 0$ is established. If there is still sliding at this instant, then it ceases after a time of the same order, and each equilibrium position $(x^*, 0)$ is therefore stable. On the other hand, if $e = 1$, then, by virtue of equality (3.2), impacts will follow periodically for an unlimited time. During flights, the x coordinate varies with a constant acceleration and relative sliding ceases at the time of impacts. This means that this coordinate increases with time without limit, which is indicative of instability.

Other impact models also exist in which the coefficient e depends on the velocity of approach.[13] In this case, the intervals between successive impacts τ_k form a decreasing sequence, the terms of which are connected by the recurrence relation

$$\tau_{k+1} = (2/Y)e(\tau_k Y/2)$$

If this sequence is summable, then the impacts decay after a finite time and the equilibrium is stable. Otherwise, it is necessary to investigate the summability of the sequence τ_k^2 which determines the change in the x coordinate accompanying a uniformly accelerating motion. It can be shown that the inequality $\tau_k^2 + \tau_2^2 + \ldots < \infty$ is the criterion of stability.

In more complex cases of impacts with a varying direction of sliding and impacts against several constraints, the calculation of the impulse is made much more difficult.[14] We shall assume that the following hypothesis is satisfied.

Hypothesis H. In the case of perturbations of the coordinates and velocities from a certain neighbourhood of the equilibrium position, all unilateral constraints develop into a stress state after a finite time which is vanishingly small together with the radius of the neighbourhood.

Proposition 5. Suppose Hypothesis H is satisfied in the case of system (1.1) with unilateral constraints. Then, for stability, it is sufficient that the conditions of one of Propositions 2–4 should be satisfied when all these constraints are replaced by bilateral constraints.

We will now consider the same examples as in the preceding section, assuming this time that the constraint is unilateral.

Examples. 1°. A body, which has a single point of contact with the support, is subjected to unilateral constraint $q = y - h(\varphi) \geq 0$. We shall assume the the function $h(\varphi)$, which describes the shape of the body, is doubly continuously differentiable and that the coefficient of restitution in formula (3.2) is less than unity. Then,

$$\dot{q} = \dot{y} - h'\dot{\varphi}, \quad \ddot{q} = \ddot{y} - h'\ddot{\varphi} - h''\dot{\varphi}^2 \tag{3.4}$$

If the regularity conditions (1.8) and (1.11) are satisfied, then the second equality of (3.4) can be represented in the neighbourhood of equilibrium in the form

$$\ddot{q} = -c + \beta(q, \dot{q}, x - x^*, \dot{\varphi}^2), \quad c > 0 \tag{3.5}$$

where the quantity β is infinitesimal. If $\beta \equiv 0$, then the time intervals τ_k between impacts form a geometrical progression with common ratio e. The existence of a second term on the right-hand side of formula (3.5) leads to the fact that the ratio τ_{k+1}/τ_k differs from e by an infinitesimal amount. In this case, the conclusion regarding the summability of these intervals is not changed and the remaining stability conditions are the same as in Example 3° of the preceding section.

2°. In the case of two points of unilateral contact, account has to be taken of the fact that $Y_{1,2} \leq 0$, since, otherwise, equilibrium is impossible. The external forces then reduce to a resultant with the projections X and $Y = Y_1 + Y_2$, the line of action of which intersects the support at the point with abscissa $\xi \in [a_1, a_2]$. We shall assume that $X \geq 0$. Equilibrium is possible if

$$X \leq -\mu Y \tag{3.6}$$

The tangential components of the reactions are denoted by T_1, T_2. The equations of motion are

$$\begin{aligned} m\ddot{x} &= X + T_1 + T_2, \quad m\ddot{y} = Y + N_1 + N_2 \\ mk^2\ddot{\varphi} &= a_1N_1 + a_2N_2 + hX + \xi Y + h(T_1 + T_2) \end{aligned} \tag{3.7}$$

In the case of sliding with retention of contact in both supports

$$\ddot{y} = 0, \quad T_1 + T_2 = -\mu(N_1 + N_2)\operatorname{sign}\dot{x}$$

which, when account is taken of condition (3.6), contradicts the first equation of (3.7). Consequently, this type of motion is impossible.

In the case of separation at the two points, the reactions are equal to zero and the normal accelerations

$$\ddot{y}_i = \ddot{y} + a_i\ddot{\varphi}, \quad i = 1, 2$$

are non-negative. When account is taken of conditions (3.7), these conditions are equivalent to the inequalities

$$k^2Y + a_i(hX + \xi Y) \geq 0, \quad i = 1, 2$$

Since $a_2 > a_1$, $X > 0$, $Y < 0$, these inequalities can be written in the equivalent form

$$k^2Y + a_1(hX + \xi Y) \geq 0, \quad a_1 > 0 \tag{3.8}$$

Then, separation at one point without sliding at the other (revolution) is impossible since $\xi \in (a_1, a_2)$. It remains to consider one further possibility: sliding (to the right) at the point C_1 with separation at the point C_2, here

$$\ddot{y}_1 = 0, \quad \ddot{y}_2 \geq 0, \quad \ddot{x}_1 > 0$$

and system (3.7) takes the form

$$\begin{aligned} &Y + N_1 + \frac{a_1}{k^2}\overline{M} = 0, \quad Y + N_1 + \frac{a_2}{k^2}\overline{M} \geq 0, \quad X - \mu N_1 + \frac{h}{k^2}\overline{M} > 0 \\ &\overline{M} = hX + \xi Y + a_1N_1 - h\mu N_1 \end{aligned} \tag{3.9}$$

Since $a_2 > a_1$, then $\bar{M} > 0$. On comparing the first and third formulae of (3.9), we arrive at the conclusion that

$$(X - \mu N)(k^2 + h^2 + a_1^2) + h(\xi - a_1)Y > 0$$

whence $X - \mu N_1 > 0$. Taking account of condition (3.6), we conclude that

$$Y + N_1 < 0, \quad a_1 > 0$$

We now write the first formula of (3.9) in the form

$$(k^2 + a_1^2 - a_1h\mu)(Y + N_1) = -a_1h(X + \mu Y) - a_1(\xi - a_1)Y \tag{3.10}$$

Since the right-hand side of equality (3.10) is positive, we arrive at the conclusion that the first factor on the left-hand side is negative. Consequently, the inequality

$$(k^2 + a_1\xi)Y + a_1hX > 0$$

is the condition for obtaining of this type of motion.

Finally, we obtain the conditions for the regularity of the equilibrium: either $a_1a_2 < 0$, that is, the projection of the centre of mass onto the support plane lies between the contact points or

$$\mu \leq (k^2 + a_1^2)/(a_1h) \tag{3.11}$$

Stability is ensured by these conditions in conjunction with the strict inequality (3.6) and Hypothesis H.

The stability conditions in the case of continual contact (Example 2°) have the same form since partial cessation of contact means that the body touches the support at one of the limit points.

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